

THE $gl(M|N)$ SUPER YANGIAN AND ITS FINITE DIMENSIONAL REPRESENTATIONS

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Abstract

Methods are developed for systematically constructing the finite dimensional irreducible representations of the super Yangian $Y(gl(M|N))$ associated with the Lie superalgebra $gl(M|N)$. It is also shown that every finite dimensional irreducible representation of $Y(gl(M|N))$ is of highest weight type, and is uniquely characterized by a highest weight. The necessary and sufficient conditions for an irrep to be finite dimensional are given.

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Running title: Representations of $Y(gl(M|N))$

1 Introduction

This is the second of a series of papers developing the representation theory of the super Yangians. In an earlier publication[1], the finite dimensional irreducible representations of the super Yangian $Y(gl(1|1))$ were classified, and explicit bases for such representations were constructed. It is the aim of this paper to carry out a similar program for the super Yangian associated with the Lie superalgebra $gl(M|N)$ for all M and N . In particular, we will prove that every finite dimensional irrep of $Y(gl(M|N))$ is of highest weight type, and uniquely characterized by a highest weight, and give the necessary and sufficient conditions for an irrep to be finite dimensional. Methods will also be developed for systematically constructing the finite dimensional irreps.

The structures and representations of the Yangians associated with ordinary Lie algebras were studied extensively[2] [3][4][6][5] [7][8]. The central result in the representation theory is Drinfeld's Theorem[2], which provides a characterization of finite

dimensional irreps in terms of highest weight polynomials, and also gives the necessary and sufficient conditions for an irrep to be finite dimensional. However, more detailed structural information on Yangian irreps, such as their dimensions and their decompositions with respect to the underlying Lie algebras, is difficult to obtain, primarily due to the existence of finite dimensional indecomposable but non - irreducible representations. At present, the understanding of the structures of irreps of Yangians is still incomplete.

Super Yangians and their quantum analogues arose naturally from the algebraic description of integrable lattice models with Lie superalgebra symmetries. Their representation theory plays a central role in the study of such models, e.g., by applying the algebraic Bethe Ansatz to diagonalize the transfer matrices. Super (quantum)Yangians are also algebraic systems of considerable mathematical interest; their structure and representations merit a thorough investigation in their own right.

Some structural features of super (quantum)Yangians, in particular, their connections with the Lie superalgebras and the related quantum supergroups, have already been studied by Nazarov[9] and also in [10]. The finite dimensional irreps of the simplest super Yangian $Y(gl(1|1))$ have been classified, and explicit bases for such irreps have been constructed[1]. It is the aim of this letter to systematically develop the representation theory of the super Yangian $Y(gl(M|N))$.

In section 2, we define the super Yangian $Y(gl(M|N))$, then study its structure. In particular, we will prove a *BPW* theorem, which will be our starting point for developing the representation theory. Section 3 contains the main results. We prove that every finite dimensional irrep of $Y(gl(M|N))$ is of highest weight type, and is uniquely characterized by the highest weight. A general method is developed to construct highest weight irreps; the necessary and sufficient conditions for an irrep to be finite dimensional are also given. In section 4 we outline another construction of the finite dimensional irreps of $Y(gl(M|N))$, which is similar to Kac' induced module construction[11] for Lie superalgebras. This construction should be useful for investigating detailed structures of irreps.

This letter relies heavily on [1]. The reasoning and techniques employed in proving some of the results of subsection 3.1 are adopted from that paper. To make this letter reasonably self contained, we spell out these proofs in some detail.

2 Structure of $Y(gl(M|N))$

2.1 Definition

The super Yangian $Y(gl(M|N))$ was first defined by Nazarov[9] following the Faddeev - Reshetikhin - Takhtajan formalism of quantization of algebraic structures. We present the definition here to fix our notation.

We will work on the complex number field \mathbf{C} . The underlying vector space of the Lie superalgebra $gl(M|N)$ is \mathbf{Z}_2 graded, with a homogeneous basis $\{E_b^a | a, b = 1, 2, \dots, M + N\}$. Introduce the gradation index $[] : \{1, 2, \dots, M + N\} \rightarrow \mathbf{Z}_2$ such that $[a] = \begin{cases} 0, & a \leq M, \\ 1, & a > M. \end{cases}$ Let $gl(M|N)_\theta$, $\theta \in \mathbf{Z}_2$, be the vector space over \mathbf{C} spanned by the E_b^a with $[a] + [b] \equiv \theta \pmod{2}$. Then $gl(M|N)_0$ and $gl(M|N)_1$ are

the even and odd subspaces of $gl(M|N)$ respectively. We will abuse the notation a bit and define $[] : gl(M|N)_0 \cup gl(M|N)_1 \rightarrow \mathbf{Z}_2$, $[x] = \begin{cases} 0, & x \in gl(M|N)_0, \\ 1, & x \in gl(M|N)_1. \end{cases}$ The Lie superalgebra $gl(M|N)$ is this \mathbf{Z}_2 graded vector space endowed with the bilinear graded bracket $[,] : gl(M|N) \otimes gl(M|N) \rightarrow gl(M|N)$,

$$[E_b^a, E_d^c] = \delta_b^c E_d^a - (-1)^{([a]+[b])([c]+[d])} E_b^c \delta_d^a. \quad (1)$$

For convenience, we can regard $gl(M|N)$ as embedded in its universal enveloping algebra. Thus the graded bracket $[,]$ can be interpreted as the graded commutator

$$[x, y] = xy - (-1)^{[x][y]}yx. \quad (2)$$

The vector module of $gl(M|N)$ is an $(M+N)$ - dimensional \mathbf{Z}_2 -graded vector space V , spanned by the homogeneous elements $\{v^a | a = 1, 2, \dots, M+N\}$ where v^a is even if $[a] = 0$ and odd otherwise. The action of $gl(M|N)$ on V is defined by $E_b^a v^c = \delta_b^c v^a$. We denote the associated vector representation of $gl(M|N)$ by π . Then in this basis $\pi(E_b^a) = e_b^a$, where $e_b^a \in End(V)$ are the standard matrix units.

Define the permutation operator $P : V \otimes V \rightarrow V \otimes V$ by

$$P(v^a \otimes v^b) = (-1)^{[a][b]} v^b \otimes v^a.$$

Then explicitly, we have

$$P = \sum_{a,b=1}^{M+N} e_b^a \otimes e_a^b (-1)^{[b]}.$$

It is well known that the following R matrix

$$R(u) = 1 + \frac{P}{u}, \quad u \in \mathbf{C}, \quad (3)$$

satisfies the graded Yang - Baxter equation.

Let us introduce

$$\begin{aligned} L(u) &= \sum_{a,b=1}^{M+N} (-1)^{[b]} t_b^a(u) \otimes e_a^b, \\ t_b^a(u) &= (-1)^{[b]} \delta_b^a + \sum_{n=1}^{\infty} t_b^a[n] u^{-n}, \end{aligned}$$

where u is an indeterminate, and the $t_b^a[n]$, $0 < n \in \mathbf{Z}_+$, are homogeneous elements of $Y(gl(M|N))$ such that $t_b^a[n]$ is even if $[a] + [b] \equiv 0 \pmod{2}$ and odd otherwise. The $L(u)$ belongs to the \mathbf{Z}_2 graded vector space $Y(gl(M|N)) \otimes End(V)$ and is even. Now $Y(gl(M|N))$ is the \mathbf{Z}_2 graded associative algebra generated by the $t_b^a[n]$, $0 < n \in \mathbf{Z}_+$, with the following defining relations

$$L_1(u) L_2(v) R_{12}(v-u) = R_{12}(v-u) L_2(v) L_1(u). \quad (4)$$

Note that equation (4) lives in $Y(gl(M|N)) \otimes End(V) \otimes End(V)$. The multiplication of the factors on both sides are defined with respect to the grading of this triple tensor

product. To gain some concrete feel about this algebra, we put (4) into a more explicit form:

$$\begin{aligned} [t_{b_1}^{a_1}(u), t_{b_2}^{a_2}(v)] &= \frac{(-1)^{\eta(a_1, b_1; a_2, b_2)}}{u - v} [t_{b_1}^{a_2}(u) t_{b_2}^{a_1}(v) - t_{b_1}^{a_2}(v) t_{b_2}^{a_1}(u)], \\ \eta(a_1, b_1; a_2, b_2) &\equiv [a_1][a_2] + [b_1]([a_1] + [a_2]) \pmod{2}; \end{aligned}$$

or equivalently,

$$\begin{aligned} [t_{b_1}^{a_1}[m], t_{b_2}^{a_2}[n]] &= \delta_{b_1}^{a_2} t_{b_2}^{a_1}[m + n - 1] - (-1)^{([a_1] + [b_1])([a_2] + [b_2])} t_{b_1}^{a_2}[m + n - 1] \delta_{b_2}^{a_1} \\ &+ (-1)^{\eta(a_1, b_1; a_2, b_2)} \sum_{r=1}^{\min(m, n)-1} \left\{ t_{b_1}^{a_2}[r] t_{b_2}^{a_1}[m + n - 1 - r] - t_{b_1}^{a_2}[m + n - 1 - r] t_{b_2}^{a_1}[r] \right\}. \end{aligned} \quad (5)$$

$Y(gl(M|N))$ admits co - algebraic structures compatible with the associative multiplication. We have the co - unit $\epsilon : Y(gl(M|N)) \rightarrow \mathbf{C}$, $t_b^a[k] \mapsto \delta_{0k} \delta_b^a (-1)^{[a]}$, the co - multiplication $\Delta : Y(gl(M|N)) \rightarrow Y(gl(M|N)) \otimes Y(gl(M|N))$, $L(u) \mapsto L(u) \otimes L(u)$, and also the antipode $S : Y(gl(M|N)) \rightarrow Y(gl(M|N))$, $L(u) \mapsto L^{-1}(u)$. Thus $Y(gl(M|N))$ is indeed a \mathbf{Z}_2 graded Hopf algebra.

Note that $Y(gl(M|N))$ also admits the following generalized tensor product structure:

$$\begin{aligned} \Delta_\alpha^{(k-1)} : Y(gl(M|N)) &\rightarrow Y(gl(M|N))^{\otimes k}, \\ L(u) &\mapsto L(u + \alpha_1) \otimes L(u + \alpha_2) \otimes \dots \otimes L(u + \alpha_k), \end{aligned} \quad (6)$$

where $\alpha_1 = 0$, and α_i , $i = 2, 3, \dots, k$, are a set of arbitrary complex parameters. Explicitly, we have

$$\begin{aligned} \Delta_\alpha^{(k-1)}(t_b^a(u)) &= \sum_{a_1, \dots, a_{k-1}} (-1)^{\sum_{i=1}^{k-1} \{([a_i] + ([a_0] + [a_i])([a_i] + [a_{i+1}])\}} \\ &\times t_b^{a_1}(u) \otimes t_{a_1}^{a_2}(u + \alpha_2) \otimes \dots \otimes t_{a_{k-1}}^a(u + \alpha_k) \end{aligned}$$

where $a_0 = b$, and $a_k = a$.

Another useful fact is the existence of an automorphism $\phi_f : Y(gl(M|N)) \rightarrow Y(gl(M|N))$ associated with each power series $f(x) = 1 + f_1 x^{-1} + f_2 x^{-2} + \dots$, which is defined by

$$t_b^a(x) \mapsto \tilde{t}_b^a(x) = f(x) t_b^a(x). \quad (7)$$

As can be easily seen, the \tilde{t}_b^a satisfy exactly the same relations as the t_b^a themselves.

Some further simple properties of the super Yangian are worth observing. Note that $Y(gl(M|N))$ as a Hopf algebra is a deformation[12] of the universal enveloping algebra of the infinite dimensional Lie superalgebra $\hat{gl}(M|N)^{(+)} = gl(M|N) \otimes \mathbf{C}[[t]]$, where $\mathbf{C}[[t]]$ denotes the ring of polynomials in the indeterminate t . Set $E_b^a[k] = E_b^a \otimes t^k$, $k \in \mathbf{Z}_+$. Then the graded bracket for $\hat{gl}(M|N)^{(+)}$ reads

$$[E_b^a[k], E_d^c[l]] = \delta_b^c E_d^a[k + l] - (-1)^{([a] + [b])([c] + [d])} E_b^c[k + l] \delta_d^a. \quad (8)$$

The universal enveloping algebra $U(\hat{gl}(M|N)^{+})$ of this Lie superalgebra is a \mathbf{Z}_2 -graded associative algebra, which may be thought as generated by $E_b^a[k]$, $k \in \mathbf{Z}_+$, subject to

the relations (8) but with the left hand side interpreted as the graded commutator defined in (2).

This algebra in fact has the structure of a \mathbf{Z}_2 -graded Hopf algebra. In particular, its co - multiplication is given by

$$\begin{aligned}\delta : U(\widehat{gl}(M|N)^{(+)}) &\rightarrow U(\widehat{gl}(M|N)^{(+)}) \otimes U(\widehat{gl}(M|N)^{(+)}), \\ E_b^a[k] &\mapsto E_b^a[k] \otimes 1 + 1 \otimes E_b^a[k].\end{aligned}$$

In order to see that $Y(gl(M|N))$ is indeed a deformation of the Hopf superalgebra $U(\widehat{gl}(M|N)^{(+)})$, we set $t_b^a[m+1] = \kappa^{-m} E_b^a[m]$, $m \in \mathbf{Z}_+$, where κ is an indeterminate. Then $Y(gl(M|N))$ is isomorphic to the algebra \tilde{U} generated by $E_b^a[m]$, $m \in \mathbf{Z}_+$, subject to the relations

$$\begin{aligned}[E_{b_1}^{a_1}[m], E_{b_2}^{a_2}[n]] &= \delta_{b_1}^{a_2} E_{b_2}^{a_1}[m+n] - (-1)^{([a_1]+[b_1])([a_2]+[b_2])} E_{b_1}^{a_2}[m+n] \delta_{b_2}^{a_1} \\ &+ \kappa \sum_{r=1}^{Min(m,n)} (-1)^{\eta(a_1, b_1; a_2, b_2)} \left\{ E_{b_1}^{a_2}[r-1] E_{b_2}^{a_1}[m+n-r] - E_{b_1}^{a_2}[m+n-r] E_{b_2}^{a_1}[r-1] \right\}.\end{aligned}$$

Regard \tilde{U} as an algebra defined on the polynomial ring $\mathbf{C}[[\kappa]]$. Then it is clear from the above equation that $U(\widehat{gl}(M|N)^{(+)}) = \tilde{U}/\kappa\tilde{U}$. Also, the co - multiplication Δ of $Y(gl(M|N))$ induces a co - associative co - multiplication $\tilde{\Delta} : \tilde{U} \rightarrow \tilde{U}$, which is clearly the deformation of the co - multiplication δ of $U(\widehat{gl}(M|N)^{(+)})$.

Important structural and representation theoretical properties of $Y(gl(M|N))$ can be obtained by investigating this Hopf superalgebra within the framework of deformation theory[12]. Results will be reported in a future publication.

The super Yangian $Y(gl(M|N))$ contains several subalgebras, which will be useful for developing the representation theory. We can easily see from the defining relations (5) that the generators $t_b^a[1]$ form the Lie superalgebra $gl(M|N)$. For each $n > 1$, the generators $t_b^a[n]$ transform as the components of an adjoint tensor operator of this $gl(M|N)$. Define a map $t_b^a[n] \mapsto \delta_{1n} E_b^a$, where E_b^a are the standard generators of $gl(M|N)$. Then it extends to an algebra homomorphism $Y(gl(M|N)) \rightarrow U(gl(M|N))$.

There also exist various Hopf (super) subalgebras of $Y(gl(M|N))$. In particular, the following will be used in the remainder of the letter:

$$\begin{aligned}Y(gl(M)) &\text{ generated by } \{t_b^a[n] \mid a, b = 1, 2, \dots, M, 0 < n \in \mathbf{Z}_+\}; \\ Y(gl(N)) &\text{ generated by } \{t_b^a[n] \mid a, b = M+1, M+2, \dots, M+N, 0 < n \in \mathbf{Z}_+\}; \\ Y(gl(1|1)) &\text{ generated by } \{t_{M+1}^M[n], t_M^{M+1}[n], t_M^M[n], t_{M+1}^{M+1}[n] \mid 0 < n \in \mathbf{Z}_+\}.\end{aligned}\quad (9)$$

Note that although both $Y(gl(M))$ and $Y(gl(N))$ are even subalgebras, together they do not form a subalgebra of $Y(gl(M|N))$. This leads to certain complications in the development of the representation theory.

2.2 BPW theorem

We now prove a version of the BPW theorem for the super Yangian $Y(gl(M|N))$, which will be of crucial importance for developing the representation theory. Let us introduce a filtration on $Y(gl(M|N))$. Define the degree of a generator $t_b^a[n]$ by $deg(t_b^a[n]) = n$,

and require that the degree of a monomial $t_{b_1}^{a_1}[n_1]t_{b_2}^{a_2}[n_2]...t_{b_k}^{a_k}[n_k]$ is $\sum_{r=1}^k n_r$. Let Y_p be the vector space over \mathbf{C} spanned by monomials of degree not greater than p . Then

$$\begin{aligned} \dots \supset Y_p \supset Y_{p-1} \supset \dots \supset Y_1 \supset Y_0 = \mathbf{C}, \\ Y_p Y_q \subset Y_{p+q}. \end{aligned}$$

Let z_1, z_2, \dots, z_k be some $t_b^a[n]$'s. Consider the product $Z = z_1 z_2 \dots z_k$, which is assumed to have $\deg(Z) = p$. It directly follows from the defining relations (5) of the super Yangian $Y(gl(M|N))$ that for any permutation σ of $(1, 2, \dots, k)$,

$$z_1 z_2 \dots z_k - \epsilon(\sigma) z_{\sigma(1)} z_{\sigma(2)} \dots z_{\sigma(k)}$$

belongs to Y_{p-1} , where $\epsilon(\sigma)$ is -1 if σ permutes the odd elements in z_1, z_2, \dots, z_k an odd number of times, and $+1$ otherwise. In particular, if $t_b^a[n]$ is odd, then $(t_b^a[n])^2 \in Y_{2n-1}$. Therefore, given any ordering of the generators $t_b^a[n]$, $0 < n \in \mathbf{Z}_+$, $a, b \in \{1, 2, \dots, M+N\}$, their ordered products of degrees less or equal to p span Y_p , where the products do not contain factors $(t_b^a[n])^2$ if $[a] + [b] \equiv 1 \pmod{2}$. It immediately follows that the ordered products of all degrees span the underlying vector space of $Y(gl(M|N))$. As we will show presently, the ordered products are also linearly independent, thus form a basis for $Y(gl(M|N))$.

Define $U_p = Y_p / Y_{p-1}$. Then the multiplication of $Y(gl(M|N))$ defines a bilinear map $U_p \otimes U_q \rightarrow U_{p+q}$, which extends to a multiplication $U \otimes U \rightarrow U$ for the space $U = \bigoplus_{p=0}^{\infty} U_p$, turning U into an associative algebra. This algebra is isomorphic to the algebra of polynomials $G[X]$ in the variables $X_b^a[n]$, $a, b \in \{1, 2, \dots, M+N\}$, $n = 1, 2, \dots$, with the isomorphism $U \cong G[X]$ defined by

$$t_b^a[n] \mapsto X_b^a[n], \quad \forall a, b, n,$$

where $X_b^a[n]$ is an ordinary indeterminate if $[a] + [b] \equiv 0 \pmod{2}$, and is a Grassmannian variable if $[a] + [b] \equiv 1 \pmod{2}$. (Note that for any Grassmannian variables ζ_1 and ζ_2 we have $\zeta_i \zeta_j = -\zeta_j \zeta_i$, $i, j = 1, 2$.) Since monomials in $X_b^a[n]$ are linearly independent as elements of $G[X]$, we conclude that ordered products of the $t_b^a[n]$ (not allowing powers of order higher than 1 of the odd $t_b^a[n]$) are linearly independent. To summarize, we have

Theorem 1 : *For any given ordering of the generators $t_b^a[n]$, $a, b \in \{1, 2, \dots, M+N\}$, $0 < n \in \mathbf{Z}_+$, the ordered products of the $t_b^a[n]$ containing no second and higher order powers of the odd generators form a basis of $Y(gl(M|N))$.*

It is useful to construct an explicit basis for $Y(gl(M|N))$. To do that, we need to fix some notations. Consider the pairs (a, b) , $a, b \in \{1, 2, \dots, M+N\}$. Let

$$\begin{aligned} \Phi_+ &= \{(a, b) | a < b\}, \\ \Phi_- &= \{(a, b) | a > b\}, \\ \Phi_0 &= \{(a, a)\}, \\ \Phi_{\pm}^{(\theta)} &= \{(a, b) \in \Phi_{\pm} | [a] + [b] \equiv \theta \pmod{2}\}. \end{aligned}$$

Given any $p = (a, b)$, we denote $\bar{p} = (b, a)$. We introduce a total ordering \succ (\prec) of all the pairs in the following way: for any $p_+ \in \Phi_+$, $p_0 \in \Phi_0$, $p_- \in \Phi_-$, we define

$p_+ \succ p_0 \succ p_-$, or equivalently, $p_- \prec p_0 \prec p_+$. For $(a, b), (c, d) \in \Phi_+$, we define $(a, b) \succ (c, d)$ (i.e., $(c, d) \prec (a, b)$) if $a < c$ or $a = c$, $b > d$; for p_1, p_2 belonging to Φ_- , $p_1 \succ p_2$ if $\bar{p}_1 \prec \bar{p}_2$; and $(a, a) \succ (b, b)$ if $a < b$.

Let

$$Q_{(a,b)}^{\{k\}}[\{n\}] = (t_b^a[n_1])^{k_1} (t_b^a[n_2])^{k_2} \dots (t_b^a[n_r])^{k_r}, \quad n_1 < n_2 < \dots < n_r,$$

$$k_1, \dots, k_r \in \begin{cases} \mathbf{Z}_+, & [a] + [b] \equiv 0 \pmod{2}, \\ \{0, 1\}, & [a] + [b] \equiv 1 \pmod{2}. \end{cases}$$

Define the ordered product

$$\prod_p^{\succ} Q_p^{\{k_p\}}[\{n_p\}],$$

which positions $Q_p^{\{k_p\}}[\{n_p\}]$ on the right of $Q_{p'}^{\{k_{p'}\}}[\{n_{p'}\}]$ if $p \succ p'$. Now

Lemma 1 *The following elements*

$$\prod_{p \in \Phi_-^{(1)}}^{\succ} Q_p^{\{k_p\}}[\{n_p\}] \prod_{q \in \Phi_-^{(0)}}^{\succ} Q_q^{\{k_q\}}[\{n_q\}] \prod_{r \in \Phi_0}^{\succ} Q_r^{\{k_r\}}[\{n_r\}] \prod_{s \in \Phi_+^{(0)}}^{\succ} Q_s^{\{k_s\}}[\{n_s\}] \prod_{t \in \Phi_+^{(1)}}^{\succ} Q_t^{\{k_t\}}[\{n_t\}], \quad (10)$$

form a basis of $Y(gl(M|N))$.

3 Finite Dimensional Irreps

We study structures of the finite dimensional irreps of $Y(gl(M|N))$ in this section. Some of the results reported here are generalizations of those on the representations of $Y(gl(1|1))[1]$ to the present case, and the proofs of these results are also adopted from [1].

3.1 Highest weight irreps

Let V be an irreducible $Y(gl(M|N))$ - module. A nonzero element $v_+^\Lambda \in V$ is called maximal if

$$t_b^a[n]v_+^\Lambda = 0, \quad \forall (a, b) \in \Phi_+, \quad n > 0,$$

$$t_a^a[n]v_+^\Lambda = \lambda_a[n]v_+^\Lambda, \quad a = 1, 2, \dots, M + N, \quad n > 0, \quad (11)$$

where $\lambda_a[n] \in \mathbf{C}$. An irreducible module is called a highest weight module if it admits a maximal vector. We define

$$\Lambda(x) = (\lambda_1(x), \lambda_2(x), \dots, \lambda_{M+N}(x)), \quad \lambda_a(x) = (-1)^{[a]} + \sum_{k>0} \lambda_a[k]x^{-k}, \quad (12)$$

and call $\Lambda(x)$ a highest weight of V .

Note that commutators amongst the $t_a^a[n]$ do not close on these generators themselves; the usual practice of using Lie's Theorem to prove the existence of a common eigenvector for them does not work here. But nevertheless, we have

Theorem 2 *Every finite dimensional irreducible $Y(gl(M|N))$ - module V contains a unique (up to scalar multiples) maximal vector v_+^Λ .*

In order to prove the theorem, we need the following

Lemma 2 Define $V_0 = \{v \in V \mid t_b^a[n]v = 0, \forall (a, b) \in \Phi_+, n > 0\}$. Then

- 1). $V_0 \neq 0$;
- 2). the generators $t_a^a[n]$ stabilize V_0 ;
- 3). for all $v \in V_0$,

$$[t_a^a[m], t_b^b[n]]v = 0, \quad \forall a, b, m, n.$$

Proof: 1). Let E be an $(M + N)$ - dimensional vector space over \mathbf{C} . We define a map $\alpha : \Phi_+ \cup \Phi_- \rightarrow E$ by setting $\alpha(a, b)$ to the vector with the a -th component being $+1$ and the b -th component being -1 , e.g., $\alpha(1, 3) = (1, 0, -1, 0, \dots, 0)$. We can introduce a partial ordering of all the elements of E by requiring that $\mu \succ \nu$ (i.e., $\nu \prec \mu$) if $\mu = \nu + \sum_{p \in \Phi_+} l_p \alpha(p)$, $l_p \in \mathbf{Z}_+$, where at least for one $p \in \Phi_+$, $l_p > 0$.

If the nonvanishing element $v \in V$ is a common eigenvector of the $t_a^a[1]$, then

$$t_a^a[1]v = \mu_a v, \quad \mu_a \in \mathbf{C}.$$

In this way, every such v is associated with a unique vector $\mu = (\mu_1, \mu_2, \dots, \mu_{M+N}) \in E$, which we call the $gl(M|N)$ weight of v . It is obvious that if a set of common eigenvectors all have different $gl(M|N)$ weights, then they must be linearly independent. Since $\{t_a^a[1] \mid a = 1, 2, \dots, M + N\}$ forms an abelian Lie algebra, it follows Lie's Theorem that there exists at least one nonvanishing element of V , which is their common eigenvector.

Let $v \in V$ be a common eigenvector of the $t_a^a[1]$ with a $gl(M|N)$ weight μ . If $v \in V_0$, we have proved the first part of the Lemma. If $v \notin V_0$, by applying $t_b^a[n]$, $(a, b) \in \Phi_+$, to v we will arrive at other common eigenvectors of the $t_a^a[1]$, which have $gl(M|N)$ weights $\succ \mu$. Since V is finite dimensional, there can only exist a finite number of vectors with $gl(M|N)$ weights $\succ \mu$. Hence repeated applications of the generators $t_b^a[n]$, $(a, b) \in \Phi_+$, to v will lead to a nonvanishing $v_0 \in V$ such that

$$\begin{aligned} t_b^a[n]v_0 &= 0, \quad \forall (a, b) \in \Phi_+, \quad n > 0, \\ t_a^a[1]v_0 &= \lambda_a[1]v_0, \quad \forall a. \end{aligned}$$

This proves that V_0 contains at least one nonzero element.

2). Let v be a vector of V_0 . We want to prove that all $t_{a_k}^{a_k}[n_k]t_{a_{k-1}}^{a_{k-1}}[n_{k-1}]\dots t_{a_1}^{a_1}[n_1]v$, $a_i = 1, 2, \dots, M + N$, $n_i > 0$, $k \geq 0$, are annihilated by $t_b^a[m]$, $(a, b) \in \Phi_+$, $m > 0$. The $k = 0$ case requires no proof. Assume that all the vectors $v_l = t_{a_l}^{a_l}[n_l]\dots t_{a_1}^{a_1}[n_1]v$, $l < k$, are in V_0 . Then

$$\begin{aligned} i). \quad & (a, b) \in \Phi_+, \quad b > c, \\ & t_b^a[m]t_c^c[n_k]v_{k-1} = -[t_c^c[n_k], t_b^a[m]]v_{k-1} \\ & = (-1)^{[c]+1} \sum_{r=0}^{\min(m, n_k)-1} (t_c^c[r]t_b^a[m+n_k-1-r] - t_c^a[m+n_k-1-r]t_b^c[r])v_{k-1} \\ & = 0, \\ ii). \quad & (a, b) \in \Phi_+, \quad b \leq c, \\ & t_b^a[m]t_c^c[n_k]v_{k-1} = [t_b^a[m], t_c^c[n_k]]v_{k-1} \\ & = \sum_{r=0}^{\min(m, n_k)-1} (t_b^c[r]t_c^a[m+n_k-1-r] - t_b^a[m+n_k-1-r]t_c^c[r])v_{k-1} \\ & \times (-1)^{[a][c]+[b]([a]+[c])} = 0. \end{aligned}$$

3). The following defining relations of $Y(gl(M|N))$

$$\begin{aligned} [t_a^a[m], t_a^a[n]] &= 0, \quad a = 1, 2, \dots, M+N, \\ [t_b^b[m], t_c^c[n]] &= (-1)^{[b]} \sum_{r=0}^{\min(m,n)-1} \left(t_b^c[r] t_c^b[m+n-1-r] - t_b^c[m+n-1-r] t_c^b[r] \right), \\ (b, c) &\in \Phi_+, \end{aligned}$$

and part 2) of the Lemma directly lead to part 3).

Proof of theorem 2: By part 3) of Lemma 2, the action of the $t_a^a[n]$ on V_0 coincides with an abelian subalgebra of $gl(V_0)$. Therefore, Lie's Theorem can be applied, and we conclude that there exists at least one common eigenvector of all the $t_a^a[n]$ in V_0 . This proves the existence of the highest weight vector. Assume that v_+ and v'_+ are two highest weight vectors of V , which are not proportional to each other. Applying $Y(gl(M|N))$ to them generates two nonzero submodules of V , which are not equal. This contradicts the irreducibility of V .

We now turn to the construction of highest weight irreps of $Y(gl(M|N))$. Let N^+ , N^- and Y^0 be the vector spaces spanned by the ordered products (as defined in (10)) of the elements $t_b^a[n]$, with $(a, b) \in \Phi^+$, $(a, b) \in \Phi^-$ and $(a, b) \in \Phi^0$ respectively. Set $Y^+ = Y^0 N^+$, and $Y^- = N^- Y^0$. We emphasize that these vector spaces do not form subalgebras of $Y(gl(M|N))$.

Consider a one dimensional vector space $\mathbf{C}v_+^\Lambda$. We define a linear action of Y^+ on it by

$$\begin{aligned} t_b^a[n]v_+^\Lambda &= 0, \quad (a, b) \in \Phi_+, \\ t_a^a[n]v_+^\Lambda &= \lambda_a[n]v_+^\Lambda, \\ t_a^a[n]y_0v_+^\Lambda &= \lambda_a[n]y_0v_+^\Lambda, \quad \forall y_0 \in Y^0. \end{aligned} \tag{13}$$

From the proof of Lemma 2 we can see that the definition (13) is consistent with the commutation relations of $Y(gl(M|N))$. Now we define the following vector space

$$\bar{V}(\Lambda) = Y(gl(M|N)) \otimes_{Y^+} v_+^\Lambda.$$

Then $\bar{V}(\Lambda)$ is a $Y(gl(M|N))$ module, which is obviously isomorphic to $N^- \otimes v_+^\Lambda$.

The action of $Y(gl(M|N))$ on this module is defined in the following way. Every vector of $\bar{V}(\Lambda)$ can be expressed as $y \otimes v_+^\Lambda$ for some $y \in N^-$. For simplicity, we write it as yv_+^Λ . Given any $u \in Y(gl(M|N))$, uy can be expressed as a linear sum of the basis elements (10). We write

$$\begin{aligned} uy &= \sum a_{\alpha,\beta} y_-^{(\alpha)} y_0^{(\beta)} + \sum b_{\mu,\nu,\sigma} y_-^{(\mu)} y_0^{(\nu)} y_+^{(\sigma)}, \\ y_-^{(\alpha)}, y_-^{(\mu)} &\in N^-, \quad y_0^{(\beta)}, y_0^{(\nu)} \in Y^0, \quad 1 \neq y_+^{(\sigma)} \in N^+. \end{aligned}$$

Then

$$u(yv_+^\Lambda) = \sum a_{\alpha,\beta} y_-^{(\alpha)} y_0^{(\beta)} v_+^\Lambda.$$

The $Y(gl(M|N))$ module $\bar{V}(\Lambda)$ is infinite dimensional. Standard arguments show that it is indecomposable, and contains a unique maximal proper submodule $M(\Lambda)$. Construct

$$V(\Lambda) = \bar{V}(\Lambda)/M(\Lambda).$$

Then $V(\Lambda)$ is an irreducible highest weight $Y(gl(M|N))$ module.

Let $V_1(\Lambda)$ and $V_2(\Lambda)$ be two irreducible $Y(gl(M|N))$ modules with the same highest weight $\Lambda(x)$. Denote by $v_{1,+}^\Lambda$ and $v_{2,+}^\Lambda$ their maximal vectors respectively. Set $W = V_1(\Lambda) \oplus V_2(\Lambda)$. Then $v_+^\Lambda = (v_{1,+}^\Lambda, v_{2,+}^\Lambda)$ is maximal, and repeated applications of $Y(gl(M|N))$ to v_+^Λ generate an $Y(gl(M|N))$ submodule $V(\Lambda)$ of W . Define the module homomorphisms $P_i : V(\Lambda) \rightarrow V_i(\Lambda)$ by

$$\begin{aligned} P_1(v_1, v_2) &= (v_1, 0), \\ P_2(v_1, v_2) &= (0, v_2), \quad v_1 \in V_1(\Lambda), \quad v_2 \in V_2(\Lambda). \end{aligned}$$

Since

$$\begin{aligned} P_1(v_{1,+}^\Lambda, v_{2,+}^\Lambda) &= (v_{1,+}^\Lambda, 0), \\ P_2(v_{1,+}^\Lambda, v_{2,+}^\Lambda) &= (0, v_{2,+}^\Lambda), \end{aligned}$$

it follows the irreducibility of $V_1(\Lambda)$ and $V_2(\Lambda)$ that $Im P_i = V_i(\Lambda)$. Now $Ker P_1$ is a submodule of $V_2(\Lambda)$. The irreducibility of $V_2(\Lambda)$ forces either $Ker P_1 = 0$ or $Ker P_1 = V_2(\Lambda)$. But the latter case is not possible, as $(0, v_{2,+}^\Lambda) \notin W$. Similarly we can show that $Ker P_2 = 0$. Hence, P_i are $Y(gl(M|N))$ module isomorphisms.

To summarize the preceding discussions, we have

Theorem 3 *Corresponding to each $\Lambda(x)$ of the form (12), there exists a unique irreducible highest weight $Y(gl(M|N))$ module $V(\Lambda)$ with highest weight $\Lambda(x)$.*

Before closing this subsection, we consider some useful facts about tensor products of highest weight irreps of $Y(gl(M|N))$. Let $W(\mu)$ and $W(\nu)$ be finite dimensional irreducible $Y(gl(M|N))$ modules with highest weights $\mu(x) = (\mu_1(x), \mu_2(x), \dots, \mu_{M+N}(x))$ and $\nu(x) = (\nu_1(x), \nu_2(x), \dots, \nu_{M+N}(x))$ respectively. (The existence of such modules will be proven in the next subsection.) Let w_+^μ and w_+^ν be the maximal vectors of these modules. Then $v_+ = w_+^\mu \otimes w_+^\nu \in W(\mu) \otimes W(\nu)$ is a maximal vector such that

$$t_a^a(x)v_+ = (-1)^{[a]}\mu_a(x)\nu_a(x)v_+, \quad \forall a.$$

Set $\lambda_a(x) = (-1)^{[a]}\mu_a(x)\nu_a(x)$, and define the \star -product of the highest weight $\mu(x)$ and $\nu(x)$ by

$$\mu(x) \star \nu(x) = (\lambda_1(x), \lambda_2(x), \dots, \lambda_{M+N}(x)).$$

Applying $Y(gl(M|N))$ to v_+ generates an indecomposable $Y(gl(M|N))$ module $\bar{V}(\mu \star \nu)$, which is contained in $W(\mu) \otimes W(\nu)$. The quotient module of $\bar{V}(\mu \star \nu)$ by its unique maximal invariant submodule yields an irreducible $Y(gl(M|N))$ module $V(\mu \star \nu)$ with highest weight $\mu(x) \star \nu(x)$, which is necessarily finite dimensional. (The maximal invariant submodule of $\bar{V}(\mu \star \nu)$ can of course be trivial. In that case, $\bar{V}(\mu \star \nu) = V(\mu \star \nu)$.) Clearly this discussion can be generalized to tensor products of more than two irreps.

3.2 Finite dimensionality conditions

Let $V(\Lambda)$ be a finite dimensional irreducible $Y(gl(M|N))$ module with highest weight $\Lambda(x)$. Denote its maximal vector by v_+^Λ . We now consider the actions of the subalgebras of $Y(gl(M|N))$ defined by (9) on the maximal vector of $V(\Lambda)$. The following $V(\Lambda)$ subspaces $Y(gl(M))v_+^\Lambda$, $Y(gl(N))v_+^\Lambda$ and $Y(gl(1|1))v_+^\Lambda$ furnish indecomposable modules over the subalgebras $Y(gl(M))$, $Y(gl(N))$, and $Y(gl(1|1))$ respectively. It is obvious but very important to note that these modules, being subspaces of $V(\Lambda)$, are finite dimensional. Thus, the necessity part of the following theorem immediately follows from Drinfeld's Theorem[2] (also see[8]) and a result of [1]:

Theorem 4 *The irreducible highest weight $Y(gl(M|N))$ - module $V(\Lambda)$ is finite dimensional if and only if its highest weight $\Lambda(x)$ satisfies the following conditions*

$$\begin{aligned} \frac{\lambda_a(x)}{\lambda_{a+1}(x)} &= \frac{P_a(x + (-1)^{[a]})}{P_a(x)}, \quad 1 \leq a < N + M, \quad a \neq M, \\ \frac{\lambda_M(x)}{\lambda_{M+1}(x)} &= \frac{\tilde{Q}_M(x)}{Q_M(x)}, \end{aligned} \quad (14)$$

where

$$\begin{aligned} P_a(x) &= \prod_{i=1}^{K_a} (x + p_a^{(i)}), \quad 1 \leq a < N + M, \quad a \neq M, \\ \tilde{Q}_M(x) &= \prod_{i=1}^{K_M} \left(1 + \frac{r_1^{(i)}}{x} \right), \\ Q_M(x) &= - \prod_{i=1}^{K_M} \left(1 + \frac{r_2^{(i)}}{x} \right), \quad \tilde{Q}_M(x), \quad Q_M(x) \text{ co-prime.} \end{aligned}$$

Proof: We only need to prove sufficiency. Let us consider the special case where the highest weight $\mu(x) = (\mu_1(x), \mu_2(x), \dots, \mu_{M+N}(x))$ of a $Y(gl(M|N))$ irrep is of the form

$$\mu_a(x) = (-1)^{[a]} + \mu_a x^{-1}, \quad \forall a. \quad (15)$$

We denote the irreducible $Y(gl(M|N))$ module with highest weight $\mu(x)$ by $W(\mu)$, and the associated irrep by π_μ . This irreducible representation can be explicitly constructed using the irreducible $gl(M|N)$ representation γ_μ with highest weight $\mu = (\mu_1, \mu_2, \dots, \mu_{M+N})$. We have

$$\pi_\mu(t_b^a(u)) = (-1)^{[b]} \delta_b^a + \gamma_\mu(E_b^a) u^{-1}, \quad \forall a, b,$$

where E_b^a are the standard $gl(M|N)$ generators.

In this case, the given conditions of the theorem are equivalent to the following constraints on μ :

$$\mu_a - \mu_{a+1} \in \mathbf{Z}_+, \quad 1 \leq a < N + M, \quad a \neq M. \quad (16)$$

This is nothing else but the necessary and sufficient conditions in order for the $gl(M|N)$ irrep γ_μ to be finite dimensional. Therefore, the $Y(gl(M|N))$ irrep π_μ is also finite

dimensional, and we have proved the sufficiency in this case. This also proves the fact that finite dimensional irreps of $Y(gl(M|N))$ indeed exist.

The next step in the proof involves showing that up to an automorphism ϕ_f of $Y(gl(M|N))$ defined by (7), every finite dimensional irreducible $Y(gl(M|N))$ module $V(\Lambda)$ with highest weight $\Lambda(x)$ sits inside the tensor product of some irreducible $Y(gl(M|N))$ modules $W(\mu)$, where the highest weights of these modules are of the form (15). To do this, we note that an automorphism ϕ_f transforms the highest weight according to $\Lambda(x) \mapsto f(x)\Lambda(x)$. Thus, it leaves the P_a , Q_M , and \tilde{Q}_M intact, but an appropriate choice of $f(x)$ will change the components of $\Lambda(x)$ into polynomials in x^{-1} defined by

$$\lambda_a(x) = \prod_{c=1}^{a-1} Q_c(x) \prod_{d=a}^{M+N-1} \tilde{Q}_d(x), \quad \forall a,$$

where we have used the following notation

$$\begin{aligned} Q_a(x) &= (-1)^{(K_a+1)[a]} \prod_{i=1}^{K_a} q_a^{(i)}(x), & q_a^{(i)}(x) &= (-1)^{[a]} \left(1 + \frac{p_a^{(i)}}{x}\right), \\ \tilde{Q}_a(x) &= (-1)^{(K_a+1)[a]} \prod_{i=1}^{K_a} \tilde{q}_a^{(i)}(x), & \tilde{q}_a^{(i)}(x) &= (-1)^{[a]} \left(1 + \frac{p_a^{(i)} + (-1)^{[a]}}{x}\right), \quad a \neq M, \\ \tilde{q}_M^{(i)}(x) &= \left(1 + \frac{r_1^{(i)}}{x}\right), & q_M^{(i)}(x) &= -\left(1 + \frac{r_2^{(i)}}{x}\right). \end{aligned}$$

Define

$$\begin{aligned} \mu^{(t,i)}(x) &= (\mu_1^{(t,i)}(x), \mu_2^{(t,i)}(x), \dots, \mu_{M+N}^{(t,i)}(x)), \\ \mu_a^{(t,i)}(x) &= \begin{cases} q_t^{(i)}(x), & t < a, \\ \tilde{q}_t^{(i)}(x), & t \geq a, \end{cases} \\ i &= 1, 2, \dots, K_t, \quad t = 1, 2, \dots, M+N-1. \end{aligned}$$

Then $\Lambda(x)$ can be expressed as the \star -product of all the $\mu^{(t,i)}(x)$

$$\Lambda(x) = \star_{t=1}^{M+N-1} \star_{i=1}^{K_t} \mu^{(t,i)}(x).$$

The $\mu^{(t,i)}(x)$ are of the form (15) and satisfy the conditions (16). Therefore, corresponding to each $\mu^{(t,i)}(x)$, there exists a finite dimensional irreducible $Y(gl(M|N))$ module $W(\mu^{(t,i)})$ with highest weight $\mu^{(t,i)}(x)$. It follows the discussions at the end of the last subsection that the tensor product module $\otimes_{t=1}^{M+N-1} \otimes_{i=1}^{K_t} W(\mu^{(t,i)})$ (The order of the tensor product is not important for us.) contains as a submodule an indecomposable $Y(gl(M|N))$ module $\bar{V}(\Lambda)$. A quotient module of $\bar{V}(\Lambda)$ gives rise to an irreducible $Y(gl(M|N))$ module which is isomorphic to $V(\Lambda)$. Being a submodule of a finite dimensional module, $\bar{V}(\Lambda)$ is finite dimensional, and so is also $V(\Lambda)$. This proves the sufficiency of the given conditions of the theorem.

4 Another Construction of Irreps

Experiences with the representation theories of the Lie superalgebras and quantum supergroups urge us to ask whether a method similar to Kac' induced module

construction[11] can be developed to build irreps of $Y(gl(M|N))$. The answer to this question is affirmative. We now outline the construction.

Introduce an auxiliary algebra $Y(gl(M)) \dot{+} Y(gl(N))$, which is the product of the two Yangians $Y(gl(M))$ and $Y(gl(N))$. In more explicit terms, $Y(gl(M)) \dot{+} Y(gl(N))$ is generated by $\{\phi_j^i[n], \psi_\nu^\mu[n] \mid i, j = 1, 2, \dots, M, \mu, \nu = M+1, M+2, \dots, M+N, 0 < n \in \mathbf{Z}_+\}$, where the $\phi_j^i[n]$ satisfy the standard defining relations of a $gl(M)$ Yangian algebra $Y(gl(M))$, and the $\psi_\nu^\mu[n]$ satisfy relations of $Y(gl(N))$, while

$$[\phi_j^i[m], \psi_\nu^\mu[n]] = 0.$$

Given a finite dimensional irreducible $Y(gl(M)) \dot{+} Y(gl(N))$ module V_0 , we define the action of the following $Y(gl(M|N))$ generators $\{t_j^i[n], t_\nu^\mu[n], t_\mu^i[n] \mid i, j = 1, 2, \dots, M, \mu, \nu = M+1, M+2, \dots, M+N, 0 < n \in \mathbf{Z}_+\}$ on V_0 by

$$\begin{aligned} t_\mu^i[n]v &= 0, \\ t_j^i[n]v &= \phi_j^i[n]v, \\ t_\nu^\mu[n]v &= \psi_\nu^\mu[n]v, \quad \forall v \in V_0. \end{aligned}$$

It follows from the first equation and the $Y(gl(M|N))$ defining relations (5) that $[t_j^i[m], t_\nu^\mu[n]]v = 0, \forall v \in V_0$. Thus this definition is self consistent.

We further define the vector space \bar{V} spanned by

$$\left\{ \prod_{p \in \Phi_-^{(1)}} Q_p^{\{k_p\}}[\{n_p\}] \otimes v \mid \forall \{k_p\}, \{n_p\}; \quad v \in V_0 \right\},$$

which furnishes an indecomposable $Y(gl(M|N))$ module, with the module action defined in the obvious way. Then the quotient module of \bar{V} by its unique maximal invariant submodule is the irreducible $Y(gl(M|N))$ module which we intend to construct. From Theorems (3) and (4) we deduce that this construction yields all the finite dimensional irreps of $Y(gl(M|N))$.

Kac' construction[11] proves to be useful for studying the representation theory of Lie superalgebras. A generalization of the method also enabled us to develop a relatively satisfactory representation theory for the type I quantum supergroups[13]. We hope that the construction presented here will also provides a practicable method for investigating detailed structures of the finite dimensional irreps of $Y(gl(M|N))$.

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